# Restrictions imposed by unitarity

Let $|\{s\}\rangle = \{s_1, s_2, \ldots, s_n\}$ where $n = [d/2]$, $s$'s are $SO(d)$ weights, represent $SO(d)$ irreducible rep. of lowest weight (i.e. scaling dimension).

We have $-iD'|\{s\}\rangle = \Delta|\{s\}\rangle$

Take highest weights of $|\{s\}\rangle$ to be $h = \{h_1, h_2, \ldots, h_n\}$.

$$[D', P'_\ell] = \left[iD, \tfrac{1}{2}(P_\ell + K_\ell) + \tfrac{i}{2}(P_\ell - K_\ell)\right]$$

$$= \frac{i}{2}[D, P_\ell] - \frac{1}{2}\{D, P_\ell]$$

$$+ \frac{i}{2}\{D, K_\ell\} + \frac{1}{2}[D, K_\ell]$$

$$= \frac{1}{2}P_\ell + \frac{i}{2}P_\ell - \frac{1}{2}K_\ell + \frac{i}{2}K_\ell$$

$$= iP'_\ell$$

$$\Rightarrow |\Delta\rangle \xrightarrow{P'_\ell} |\Delta+1\rangle \xrightarrow{P'_m} |\Delta+2\rangle \cdots$$

On the other hand

$$[D', K'_\ell] = -iK_\ell$$

$$\Rightarrow 0 \xleftarrow{K'_\ell} |\Delta\rangle \xleftarrow{K'_m} |\Delta+1\rangle \cdots$$

Using the fact that $P'^{\dagger} = K'$ the requirement that states must have positive norm (unitarity) translates into

$$A_{\mu\{s\},\nu\{t\}} = \langle\{s\}|K'_\mu P'_\nu|\{t\}\rangle \geq 0$$

Using the commutator

$$[K'_\mu, P'_\nu] = -2i\,(D'\delta_{\mu\nu} + M'_{\mu\nu})$$

we get

$$A_{\mu\{s\},\nu\{t\}} = 2\langle\{s\}|\,\Delta_0 + (-i)M'_{\mu\nu}\,|\{t\}\rangle$$

$\Rightarrow$ Positivity of $A$ is equivalent to the condition that the matrix

$$B_{\mu\{s\},\nu\{t\}} = \langle\{s\}|(-i)M'_{\mu\nu}|\{t\}\rangle \qquad (*)$$

has no eigenvalues smaller than $-\Delta_0$.

Now notice

$$(-i)\,M'_{\mu\nu} = \frac{1}{2}(-i)(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha})\,M'^{R}_{\alpha\beta}$$

That is

$$M'_{\mu\nu} = (V \underset{\overset{\uparrow}{\text{tensor-product}}}{\cdot} M'^{R})_{\mu\nu}$$

where $(V_{\alpha\beta})_{\mu\nu} = (-i)(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha})$ are $SO(d)$ generators in the vector-representation.

$$\Rightarrow \quad B_{\mu\{s\},\nu\{t\}} = (V\cdot M'^{R})_{\mu\{s\},\nu\{t\}}$$

Now

$$V\cdot M' = \frac{1}{2}\big((V+M')\cdot(V+M') - V\cdot V - M'\cdot M'\big)$$

This is similar to the spin-orbit interaction in QM: $L^i\cdot S^i = \frac{1}{2}[(L+S)^2 - L^2 - S^2]$

The operators $S^2$ and $L^2$ have eigenvalues $s(s+1)/2$ and $\ell(\ell+1)/2$ and $(L+S)^2$ has eigenvalues in the tensor product rep.

$\ell \otimes s \longrightarrow j(j+1)/2\,,\ j = |\ell - s|, \dots, \ell+s$

Let $M'$ transform in representation $R$, then condition $(*)$ becomes

$$\Delta_0 \geq \frac{1}{2}\Big(C_2(R) + C_2(V) - C_2(R')\Big)$$

where $R'$ is the representation with the smallest Casimir in the expansion of $R \otimes V$.

Recall

$$C_2(\{h\}) = \sum_{i=1}^{[d/2]} \left(h_i^2 + (d-2i)h_i\right)$$

For example, for the vector-rep in $d=3$ we get

$$\{h\}_{vec} = (\ell)$$

$$\Rightarrow \ C_2(\{h\}_{vec}) = \ell^2 + (3-2)\ell = \ell(\ell+1)$$

special cases: $C_2(\text{scalar}) = 0$, $C_2(\text{spinor}) = (d/8)(d-1)$,

$C_2(\text{vector}) = d-1$

Take $R = \{h_1, h_2, \dots, h_{[d/2]}\}$

$$\rightarrow V_{\ell=1} \cdot R = \{h_1 \pm 1, h_2, \dots, h_{[d/2]}\} + \{h_1, h_2 \pm 1, \dots\} + \dots + \{h_1, h_2, \dots, h_{[d/2]} \pm 1\}$$

with the restriction that $\{h_1', h_2', \dots, h_{[d/2]}'\}$ is non-zero iff $h_1' \geq h_2' \geq h_3' \geq \dots \geq h_{[d/2]}' \geq 0$

Thus the condition $\Delta_0 \geq \frac{1}{2}\left(c_2(R)+c_2(V)-c_2(R')\right)$

becomes:

$$\Delta_0 \geq \frac{1}{2}\left(\sum_{i=1}^{[d/2]}\left(h_i^2+(d-2i)h_i\right)+d-1-\sum_{i=1}^{[d/2]}\left(h_i'^2+(d-2i)h_i'\right)\right)$$

with $\{h'\} = \{h_1, h_2, \dots, h_j-1, h_{j+1}, \dots, h_{[d/2]}\}$

$$\overset{!}{=} |h_j| + d - j - 1$$

To maximize the rhs, choose $j$ to be the smallest value s.t. $h_j \geq |h_{j+1}|+1$

If no such $j$ exists → all $h_i$ are equal:

1) if $h_i = 0 \ \forall i$ : $\Delta_0 \geq 0$
2) if $h_i \geq 1 \ \forall i$ : choose $i = [d/2]$
3) if $h_i = \frac{1}{2} \ \forall i$ (spinor rep) : $\Delta_0 \geq (d-1)/2$
($i = [d/2]$)

Let's consider some examples:

1) $\underline{d=3}$:

SO(3) reps are given by a choice of $j \in \frac{1}{2}\mathbb{Z}$

We have

$$\Delta_0 \geq 0 \quad (j=0)$$
$$\Delta_0 \geq 1 \quad (j=\tfrac{1}{2})$$
$$\Delta_0 \geq j+1 \quad (j \geq 1)$$

2) <u>$d=4$:</u>

$SO(4) = SU(2)\times SU(2)$

→ reps are labeled by $j_1, j_2 \in \frac{1}{2}\mathbb{Z}$

We have

$$\Delta_0 \geq f(j_1) + f(j_2)$$

where

$$f(j) = 0 \text{ for } j=0, \quad f(j) = j+1 \text{ for } j>0$$

3) <u>$d=5$:</u>

$SO(5)$ reps are labeled by highest weight $h_1, h_2$

$$\Delta_0 \geq 0 \quad (h_1 = h_2 = 0)$$

$$\Delta_0 \geq 2 \quad (h_1 = h_2 = \tfrac{1}{2})$$

$$\Delta_0 \geq h+2 \quad (h_1 = h_2 = h \neq 0, \tfrac{1}{2})$$

$$\Delta_0 \geq h_1 + 3 \quad (h_1 > h_2)$$

4) <u>$d=6$:</u>

$SO(6)$ highest weights are $h_1, h_2, h_3$

We have

$$\Delta_0 \geq 0 \quad (h_1 = h_2 = h_3 = 0)$$

$$\Delta_0 \geq h+2 \,(h_1 = h_2 = |h_3| = h \neq 0)$$

$$\Delta_0 \geq h+3 \,(h_1 = h_2 > |h_3|)$$

$$\Delta_0 \geq h_1 + 4 \,(h_1 > h_2)$$

In arbitrary dimension d special reps obey

$$\Delta_0 \geq 0 \quad \text{(scalar)}$$

$$\Delta_0 \geq (d-1)/2 \quad \text{(spinor)}$$

$$\Delta_0 \geq (d-1) \quad \text{(vector)}$$

## Free conformally invariant fields

Why do operators in a CFT appear in multiplets of SO(d)? For example, we know that mass-less particles transform in SO(d-2)

Let's try to understand this:

A unitary irreducible representation of the conformal algebra, denoted by $R^{CFT}$, can be decomposed as a direct sum

$$R^{CFT} = \bigoplus m_n R_n^{SO(d)} \otimes R_n^{SO(d)} \quad \text{"conformal module"}$$

for $SO(d) \otimes SO(d) \subset SO(d,2)$

The SO(d) here is generated by $M'$, and the SO(2) by $D'$

- → $m_n$ is a positive integer
- → $D'$ eigenvalues are of the form $\Delta_0 + n$ with $n \in \mathbb{Z}_+$, $m_0 = 1$

$\Delta_0$ is lowest weight of the module
"scaling dimension"
Let the rep $R_0^{SO(d)}$ have lowest weights
$\{h_1, \ldots, h_{[d/2]}\}$
→ full conformal module is determined by $\Delta_0$ and $\{h\}$

Consider now a multiplet of operators $\phi_\alpha(x)$
→ $\phi_\alpha(x) = e^{-iP\cdot x}\phi_\alpha(0)\, e^{iP\cdot x}$ (in Euclidean theory $P \to P'$)

We have
$$[P_\mu, \phi_\alpha(x)] = (-i)\left(-\partial_\mu \phi_\alpha(x)\right)$$
$$[M_{\mu\nu}, \phi_\alpha(0)] = (-M^R_{\mu\nu})_\alpha^{\ \beta}\, \phi_\beta(0)$$
$$[D, \phi_\alpha(0)] = \Delta_0\, \phi_\alpha(0)$$

How does $SO(d-2)$ appear?
Let's restrict to $P\cdot P\, \phi_\alpha(x) = 0 \quad \forall \phi$
→ conformal transformation on $\phi$ yields
$$P\cdot P\, T_a\, \phi_\alpha(x) = 0 \quad \forall\, T_a \in SO(d,2)$$
$$\rightarrow \quad [P\cdot P, T_a]\, \phi_\alpha(x) = 0$$
Set $T_a = K_\mu$ gives
$$P_\mu(-iM^R_{\mu\nu})_\alpha^{\ \beta}\, \phi_\beta = \left(\frac{d-2}{2} - \varepsilon_0\right) P_\nu\, \phi_\alpha \quad (**)$$
(d equations)

Go to the frame $p_\mu = (1,1,0,\cdots,0)$

For $\nu = 2,3,\dots,d-1$:

$$(M^R_{1i} - M^R_{0i})\phi = 0$$

M's are SO(d-1,1) matrices

go to SO(d) $\longrightarrow$ $(M^R_{1\ell} + i M^R_{0\ell})\phi = 0$

consider the su(2)-sub-algebra of SO(d) generated by

$$M^R_{01}, \quad M^R_{1\ell} + i M^R_{0\ell}, \quad M^R_{1\ell} - i M^R_{0\ell}$$

Then (* *) for $\nu > 1$ says that

$\phi$ is of highest weight, but 2nd to $[d/2]^{th}$ weights can take arbitrary values

From $\nu = 0$ and 1 we get from (**)

$$\Delta_0 = h_1 + \frac{d-2}{2}$$

($h_1$ highest 1st weight of $\phi$)

→ SO(d) weight connected with rotations in "momentum-time" plane is restricted to be highest weight, remaining weights fill out a SO(d-2) rep.

→ SO(d) scalar is SO(d-2) scalar
SO(d) spinor is SO(d-2) spinor
SO(d) vector is SO(d-2) scalar
SO(d) anti-sym tensor is SO(d-2) vector